SUFFICIENT CONDITIONS FOR IMAGINARY ROOTS OF ALGEBRAIC EQUATIONS

By Otto Dunkel

Introduction. The study of algebraic equations has led to the development of a considerable number of conditions, necessary or sufficient or both, for the existence of imaginary roots of such equations. The theorems of Descartes, Budan-Fourier, Sturm, Newton-Sylvester,* which afford such conditions, are well known.

More recently Professor E. B. Van Vleck † has given a sufficient condition for the existence of the maximum number of imaginary roots; and Professor Kellogg ‡ has given a necessary condition for the reality of all the roots, which is allied to a special case of a more general theorem which it is the purpose of this paper to develop. It will be shown that several well known conditions can be easily derived from this general theorem, which we shall refer to as the Fundamental Theorem; and in some instances sharper tests than those usually given can be obtained. One of the special cases of this theorem will be shown to be part of Newton's theorem on algebraic equations. The theorem itself permits not only of a generalization of some tests already known but also of the derivation of new tests.

Fundamental Theorem. The general equation of the n^{th} degree:

$$(1) \qquad f(x) \equiv p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0$$

will be considered; but for simplicity in the statement and also in the proof of the theorem it will be convenient to write the equation in the form:

$$(1') \quad a_0 x^n + n a_1 x^{n-1} + \cdots + \frac{n!}{(n-i)!\,i!} a_i x^{n-i} + \cdots + n a_{n-1} x + a_n = 0,$$

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* Cf. E. Netto, *Vorlesungen über Algebra*, vol. 1, for the statement of these theorems: Descartes' Theorem, p. 219; Budan-Fourier's Theorem, p. 216; Sturm's Theorem, p. 238; Sylvester's Theorem, p. 225; Newton's Theorem, p. 233.

† Van Vleck, Annals of Mathematics, ser. 2, vol. 4 (1903), p. 191.

‡ O. D. Kellogg, Annals of Mathematics, ser. 2, vol. 9 (1908), p. 97.

where

$$a_i = \frac{(n-i)!\,i!}{n!} p_i, \qquad i = 0, 1, 2, \cdots n.$$

The series of numbers $a_0, a_1, a_2, \cdots a_n$ computed from the coefficients of (1) by means of the relation in (1′) will be referred to as the *binomial coefficients* of the equation (1).

The Fundamental Theorem which we shall now prove may be stated as follows:

If $a_0, a_1, a_2, \cdots a_n$ are the binomial coefficients of an equation of the n^{th} degree; and, if any equation of lower degree whose binomial coefficients are a consecutive set of the given binomial coefficients, e. g.:

$$(2) \quad a_m x^t + t a_{m+1} x^{t-1} + \cdots + \frac{t!}{(t-i)!\,i!} a_{m+i} x^{t-i} + \cdots + t a_{m+t-1} x + a_{m+t} = 0,$$

has imaginary roots; then the original equation has at least as many imaginary roots.

The proof of this theorem follows very easily from two theorems which we shall state as lemmas.

First Lemma. If any derived equation of the given equation (1), i. e.,

$$f^i(x) \equiv \frac{d^i f(x)}{dx^i} = 0,$$

has imaginary roots, the original equation must have at least as many. The truth of this is very easily seen from a figure of the curve $y = f(x)$ by the application of Rolle's Theorem.*

Second Lemma. If any equation has imaginary roots, its reciprocal equation has just as many; and the converse is also true. Since the reciprocal equation is obtained from the original equation by the substitution $x = 1/y$, and its roots are consequently the reciprocals of the roots of the original equation, the truth of this second lemma is seen at once.

The $(n-m-t)^{th}$ derived equation of (1′) can be written down immediately, since the result of the differentiation will be of the $(m+t)^{th}$ degree

* Netto, *Algebra*, vol. 1, p. 208. The theorem is not stated explicitly by Netto in the above form, but follows from his Satz II. The analytical proof is easy.

with corresponding binomial coefficients, after dividing out a factor common to each term of the result;* i. e.,

$$(3)\quad a_0x^{m+t} + (m+t)\,a_1x^{m+t-1} + \cdots + (m+t)\,a_{m+t-1}x + a_{m+t} = 0.$$

The reciprocal equation of (3) is

$$(4)\quad a_{m+t}y^{m+t} + (m+t)\,a_{m+t-1}y^{m+t-1} + \cdots + (m+t)\,a_1y + a_0 = 0.$$

The m^{th} derived equation of (4) is of degree t; and, if we neglect again a certain factor, it can be written:

$$(5)\qquad a_{m+t}y^t + ta_{m+t-1}y^{t-1} + \cdots + ta_{m+1}y + a_m = 0.$$

Finally the reciprocal equation of this last (5) is

$$(6)\qquad a_mx^t + ta_{m+1}x^{t-1} + \cdots + ta_{m+t-1}x + a_{m+t} = 0.$$

Suppose now (6) has $2k$ imaginary roots; then by the second lemma, (5) has $2k$; and in turn by the first lemma, (4) has at least $2k$; then (3) has at least $2k$; and finally by the same reasoning the original equation (1′) has at least $2k$ imaginary roots and hence our fundamental theorem is proved.

Specializations of the Fundamental Theorem. In order to obtain special sufficient conditions for the existence of imaginary roots of (1) or (1′), we have merely to write down any selected number of consecutive binomial coefficients, form the corresponding equation (with binomial factors), and impose the condition that it have imaginary roots. We shall thus obtain a set of relations such that if any one is satisfied the original equation has imaginary roots.

The simplest set of conditions of this kind is that in which three consecutive coefficients are chosen and quadratic equations are formed. For an equation of the n^{th} degree there are $(n-1)$ such quadratics of the type:

$$(7)\qquad a_{i-1}x^2 + 2a_ix + a_{i+1} = 0, \qquad i = 1, 2, 3 \cdots n-1.$$

If now for any one of these equations

$$(8)\qquad a_i^2 - a_{i-1}a_{i+1} < 0,$$

equation (7), and consequently (1′), will have at least one pair of imaginary

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* Burnside and Panton, *Theory of Equations*, vol. 1, p. 68. The differentiation of the general term of (1′) with a subsequent slight change in the form of the numerical factor shows this at once.

roots. If we return to the notation of (1) by means of the relation between the a's and p's as given in (1′), we shall have the result:

If any one of the following inequalities is satisfied:

$$(8') \qquad p_i^2 - \frac{i+1}{i}\,\frac{n-i+1}{n-i}\,p_{i-1}p_{i+1} < 0, \qquad i = 1, 2, 3, \cdots n-1,$$

the equation (1) *has at least one pair of imaginary roots.**

This result may also be stated in the following form:

If the equation (1) *has all its roots real, then the following* $n-1$ *inequalities must hold*:

$$(8'') \qquad p_i^2 - \frac{i+1}{i}\,\frac{n-i+1}{n-i}\,p_{i-1}p_{i+1} \geqq 0, \qquad i = 1, 2, 3, \cdots n-1.$$

The condition given by Professor Kellogg† corresponding to (8′) would read: if for any value of i

$$p_i^2 - p_{i-1}p_{i+1} \leqq 0, \qquad i = 1, 2, 3, \cdots n-1,$$

then there are imaginary roots. In the cases in which either test applies, i. e., when p_{i-1} and p_{i+1} have the same sign, it is seen that (8′) gives a somewhat sharper test.

This method of obtaining sufficient conditions for imaginary roots can be extended. We could for example make use of a series of cubics:

$$(9) \qquad a_i x^3 + 3a_{i+1}x^2 + 3a_{i+2}x + a_{i+3} = 0,$$

and, by imposing the condition that these cubics have imaginary roots, we would obtain a series of $n-2$ inequalities such that, if any one is satisfied, the original equation has imaginary roots.

Relation to the Newton-Sylvester Theorem. The condition (8′) is a special case of the theorem of Newton previously referred to. Newton's theorem may be stated as follows:

From the coefficients of the equation (1) the following sequence of $(n+1)$ numbers is formed:

$$(10) \quad p_0^2, \quad p_1^2 - l_1 p_0 p_2, \quad p_2^2 - l_2 p_1 p_3, \quad \cdots \quad p_i^2 - l_i p_{i-1} p_{i+1}, \quad \cdots \quad p_n^2,$$

* The case in which three consecutive coefficients are zero will be considered later; and it will be shown that in this case there are imaginary roots.

† Kellogg, loc. cit.

where

$$l_i = \frac{i+1}{i}\,\frac{n-i+1}{n-i}\;;$$

then the number of real roots of (1) is not greater than the number of permanences of sign in the sequence (10), the difference being even; and therefore the number of imaginary roots is not less than the number of variations of sign in the same sequence. In particular, if any one member of the sequence is negative, the equation has imaginary roots; and this is precisely the result (8′). If all the roots of the equation are real, then the members of (10) must be either positive or zero.

This theorem was given by Newton* without proof and remained without proof until Sylvester† proved a more general theorem, from which Newton's theorem was obtained as a specialization, just as Descartes' Rule of Signs can be obtained as a specialization of the theorem of Budan-Fourier.

Derivation of Other Special Tests. A well known theorem in regard to the roots of an equation with one or more zero coefficients can be easily derived from the Fundamental Theorem.

Suppose in the equation (1)

$$(11) \qquad p_{m+1} = p_{m+2} = \cdots = p_{m+t} = 0,$$

while $p_m \neq 0$ and $p_{m+t+1} \neq 0$; then the equation corresponding to (2) takes the form:

$$a_m x^{t+1} + a_{m+t+1} = 0.$$

If t is even, this equation has t imaginary roots; if t is odd, it has $t+1$ or $t-1$ according as a_m and a_{m+t+1} have like or unlike signs, or according as p_m and p_{m+t+1} have like or unlike signs.

Hence, by the Fundamental Theorem, *if any equation have t successive coefficients zero, there are at least t imaginary roots if t is an even number; if t is odd, there are at least $t+1$ or $t-1$ according as the coefficients just before and after those vanishing have like or unlike signs.*‡

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* I. Newton, *Arithmetica universalis*, Cambridge, 1707; second edition, London, 1722, chap. II.

† J. J. Sylvester, *Phil. Mag.* ser. 4, vol. 31 (1866), p. 214. For a proof of this theorem see E. Netto, loc. cit., vol. 1, pp. 225–235.

‡ Cf. Burnside and Panton, *Theory of Equations*, vol. 1, p. 197, where this result is deduced from the Budan-Fourier Theorem; also Netto, *Algebra*, vol. 1, p. 221.

This result can now be applied in turn to the original equation (1) or to any one of the equations (2) derived from (1) in order to obtain other forms of sufficient conditions for imaginary roots. As an example of this method, let us multiply the left-hand side of (1) by the left-hand side of

$$x^r + c_1 x^{r-1} + c_2 x^{r-2} + \cdots + c_{r-1} x + c_r = 0. \tag{12}$$

We shall thus obtain an equation of degree $(n + r)$, which has the same number of imaginary roots as (1) provided that all the roots of (12) are real. Hence if we can choose a set of c's so that all the roots of (12) are real and also so that two successive coefficients of the resulting equation are zero, i. e., so that for a special value of i,

$$\begin{aligned} c_r p_i + c_{r-1} p_{i+1} + \cdots + c_1 p_{i+r-1} + p_{i+r} = 0, \\ c_r p_{i+1} + c_{r-1} p_{i+2} + \cdots + c_1 p_{i+r} + p_{i+r+1} = 0; \end{aligned} \tag{13}$$

then the new equation, by the theorem above, has imaginary roots; and consequently the equation (1) must have imaginary roots.

For $r = 1$, all of these conditions are satisfied if

$$\begin{vmatrix} p_i & p_{i+1} \\ p_{i+1} & p_{i+2} \end{vmatrix} = 0;$$

hence:

*If three consecutive coefficients of an equation are in geometric progression, the equation has imaginary roots.**

For $r = 2$, (12) and (13) lead to the result of Professor Kellogg † that if the quadratic

$$\begin{vmatrix} x^2 & x & 1 \\ p_{i+2} & p_{i+1} & p_i \\ p_{i+3} & p_{i+2} & p_{i+1} \end{vmatrix} = 0 \tag{14}$$

has real roots, the equation (1) has imaginary roots.

The special case in which (12) becomes $(x - 1)^2$, and (13) becomes

$$\begin{aligned} p_i - 2p_{i+1} + p_{i+2} = 0, \\ p_{i+1} - 2p_{i+2} + p_{i+3} = 0, \end{aligned}$$

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* This also follows directly from (8′), which is satisfied whenever the three coefficients are in geometrical progression.

†Kellogg, loc. cit.

gives the theorem of Hermite*: *If four successive coefficients of an equation are in arithmetic progression, the equation has imaginary roots.*

A result analogous to (14) may be obtained in a different way. Consider any one of the equations (2) and replace x by $y+h$ and apply to the result the condition (8); for simplicity, consider the cubic (9) which becomes after the substitution:

$$(15)\qquad a_i y^3 + 3\phi_1(h)\, y^2 + 3\phi_2(h)\, y + \phi_3(h) = 0,$$

where

$$\phi_j(h) = a_i h^j + j a_{i+1} h^{j-1} + \frac{j(j-1)}{2!} a_{i+2} h^{j-2} + \cdots .\dagger$$

If we can find a real value of h so that

$$\begin{vmatrix} \phi_2(h) & \phi_1(h) \\ \phi_3(h) & \phi_2(h) \end{vmatrix} < 0,$$

then by (8) the equation (15) has imaginary roots and therefore equation (1′) has also imaginary roots. After a simple reduction this inequality becomes:

$$(16)\qquad \begin{vmatrix} a_{i+1}h + a_{i+2} & a_i h + a_{i+1} \\ a_{i+2}h + a_{i+3} & a_{i+1} h + a_{i+2} \end{vmatrix} = \begin{vmatrix} h^2 & -h & 1 \\ a_{i+2} & a_{i+1} & a_i \\ a_{i+3} & a_{i+2} & a_{i+1} \end{vmatrix} < 0.$$

This inequality can be satisfied if the determinant on the left has real and distinct roots, i. e., if

$$(17)\qquad \begin{vmatrix} a_{i+2} & a_i \\ a_{i+3} & a_{i+1} \end{vmatrix}^2 - 4 \begin{vmatrix} a_{i+2} & a_{i+1} \\ a_{i+3} & a_{i+2} \end{vmatrix} \cdot \begin{vmatrix} a_{i+1} & a_i \\ a_{i+2} & a_{i+1} \end{vmatrix} > 0. \ddagger$$

If then (17) *is true, the original equation has imaginary roots.*

* E. Netto, loc. cit., p. 225. This theorem also follows from (14).

† See Burnside and Panton, *Theory of Equations*, vol. 1, p. 68.

‡ This reduces to

$$\frac{1}{a_i^2}\Big[\big(a_{i+3}a_i^2 - 3a_i a_{i+1} a_{i+2} + 2a_{i+1}^3\big)^2 + 4\big(a_i a_{i+2} - a_{i+1}^2\big)^3\Big] > 0,$$

in which the square brackets enclose the discriminant of the cubic (9). This then is the best test of this kind which it is possible to obtain.

Combination of Van Vleck's Method with the Fundamental Theorem. A method similar to that employed by Professor Van Vleck * can be used to derive from the Fundamental Theorem another form of sufficient condition for imaginary roots.

Given an algebraic equation of degree k, written in the form (1); then if k is even and greater than two, and if

$$(18)\quad q_1^2 - 2q_0 q_2 < 0, \qquad q_{k-1}^2 - 2q_{k-2}\, q_k < 0, \qquad q_i^2 - q_{i-1}\, q_{i+1} < 0,$$
$$i = 3, 5, 7, \cdots k-3,$$

all its roots are imaginary.† For from (18) it is clear that none of the coefficients with even subscripts can be zero and also that they must have the same sign.

Writing the equation in the following way:

$$(19)\quad x^{k-2}\left(q_0x^2 + q_1x + \frac{q_2}{2}\right) + \frac{x^{k-4}}{2}\left(q_2x^2 + 2q_3x + q_4\right)$$
$$+ \frac{x^{k-6}}{2}\left(q_4x^2 + 2q_5x + q_6\right) + \cdots + \frac{x^2}{2}\left(q_{k-4}x^2 + 2q_{k-3}x + q_{k-2}\right)$$
$$+ \left(\frac{q_{k-2}}{2}x^2 + q_{k-1}x + q_k\right) = 0,$$

it is seen that the expression can never vanish for real values of x; since from (18) the quadratic expressions in the brackets cannot vanish for real values of x and for such values have the same sign (that of the coefficients of even subscripts).

Let us consider now one of the equations (2) derived from the original equation written with binomial coefficients, supposing t to be even. If we apply the result above to this equation, we shall obtain the following result in regard to the roots of $(1')$:

If any consecutive set of coefficients of the equation $(1')$ satisfy the following inequalities:

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* Loc. cit.

†For $k = 2$, replace condition (18) by $q_1^2 - 4q_0q_2 < 0$.

$$a^2_{m+1} - \frac{t-1}{t} a_m a_{m+2} < 0,$$

(20) $$a^2_{m+i} - \frac{i}{i+1}\,\frac{t-i}{t-i+1}\, a_{m+i-1}a_{m+i+1} < 0,$$

$$a^2_{m+t-1} - \frac{t-1}{t} a_{m+t}a_{m+t-2} < 0, \qquad i = 3, 5, 7, \cdots t-3,$$

then the equation (1′) has at least t imaginary roots.*

This would not afford a convenient test, owing to the dependence of the numerical coefficients upon the number of inequalities in (20). This difficulty can be removed by a slight sacrifice of sharpness in the test; for in no case will any of these numerical coefficients be so small as $\frac{1}{2}$.† We have then the following result:

If any consecutive set of coefficients of the equation (1′) *satisfy the following inequalities:*

(21) $$a^2_{m+i} - \tfrac{1}{2}\, a_{m+i-1}\, a_{m+i+1} < 0, \quad i = 1, 3, 5, \cdots t-1,$$

the equation (1′) ***has at least t imaginary roots.***

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*For $t = 2$, replace (20) by $a^2_{m+1} - a_m a_{m+2} < 0$.

† In fact they are greater than or equal to 9/16 and this fraction could be used in the inequality (21) instead of the $\frac{1}{2}$.